Review Session – Test 2

7–9 PM

1. Derivatives

1. $y = x^a \longrightarrow y' = ax^{a-1}$

Memorize all the simple derivatives

Example 1:

$f(z) = \frac{3e^{z}}{11} - \frac{e}{3}$

$f'(z) = e^{z}\left(\frac{3}{11}\right) - 0 = \frac{3e^{z}}{11}$

$h(x) = \underline{e^{4z}} + \underline{\pi}$ constants

$h'(x) = ?\ 0$

equation of tangent lines

$y = mx + b$

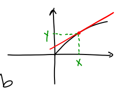
$\hookrightarrow$ slope $= y'$

ex: $f(x) = \sqrt{x}$ at $x = 4$

(1) $f(x) = x^{1/2} \longrightarrow f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$y = \frac{1}{2\sqrt{x}}x + b$

$y(4) = \sqrt{4} = 2$



$2 = \frac{1}{2\sqrt{x}}x + b$

$2 = \frac{1}{4}x + b \longrightarrow b = 1$

$\boxed{y = \frac{1}{4}x + 1}$

ex: $F(x) = \frac{1+x}{(2+x)(3+x)}$

$F'(x) = ?$

$$\frac{(2+x)(3+x)) - (1+x)((1)(3+x) + (1)(2+x))}{((2+x)(3+x))^2}$$

Trig. functions:

ex: $h(\theta) = \sin(\theta)\cos(\theta)$

$h'(\theta) = \cos(\theta)\cos(\theta) + \sin(\theta)(-\sin(\theta))$

$= \cos^2\theta - \sin^2\theta$

ex: $y = (\arctan(x^2) + 1)^3$

$y' = 3(\arctan(x^2)+1)^2\left(\frac{1}{1+(x^2)^2}\right)(2x)$

Implicit diff.

ex: $\cos x \sin y = 2 - \sin x$

(1) $\cos x \sin y - 2 + \sin x = 0$

(2) $(-\sin x)(\sin y) + (\cos y)(\cos x)(y')$

$- 0 + \cos x = 0$

$\cos y \cos x (y') = \sin x \sin y - \cos x$

$y' = \frac{\sin x \sin y - \cos x}{\cos y \cos x}$

ex: $y = 3x + 1 \longrightarrow y' = 3$

diff. $3y^2 + 2x = 0$

$6y + 2 = 0$ ✗

$6y(y') + 2 = 0$

Second derivative

(1) $f(x)$, $h(z)$, $g(\theta)$

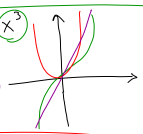
(2) $f'(x)$, $h'(z)$, $g'(\theta)$ rate of change or slope

(3) $f''(x)$, $h''(z)$, $g''(\theta)$ concave up or down

Ex: $y = x^3$

(1) $y' = 3x^2$

(2) $y'' = 6x$



Second derivatives

(+) ∪

(−) ∩

ex: $f(x) = x^2 \ln x$

1st and 2nd derivative and critical points

(1) $f'(x) = (2x)(\ln x) + (\cancel{x^2})\left(\frac{1}{\cancel{x}}\right)$

(2) Simplify:

$f'(x) = x\underbrace{(2\ln x + 1)}_{\text{solve for it}}$

$\hookrightarrow x = 0$

$2\ln x + 1 = 0 \longrightarrow \ln x = -\frac{1}{2}$

$x = e^{-1/2}$

Domain of the main function:

$x \in (0, \infty)$

0 — $e^{-1/2}$ + $f'(x)$
↘ ↗ $f(x)$

if not in the domain of main function
→ not a critical point

$e^{-1/2}$ is a local min.

② $2x\ln x + x$

$y'' = 2(\ln x) + (2x)(\frac{1}{x}) + 1$

$= 2\ln x + 2 + 1 = 2\ln x + 3$

Taylor Approximations

$$\boxed{T_n(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \dots + \frac{f^n(a)(x-a)^n}{n!}}$$

a → base point

$n! = 1 \times 2 \times 3 \times \dots \times n$

$T_1(x)$ → linear Approx. $L(x)$

$T_2(x)$ → Quadratic Approx. $Q(x)$

$g(x) = 4x^4 + x^3 + 4x^2 + 3x + 1$
x near 0

$T_2(x) = ?$

① write down the formula for $T_2(x)$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2}$

$g'(x) = 16x^3 + 3x^2 + 8x + 3$

$g'(0) = 3$

$g''(x) = 48x^2 + 6x + 8$

$g''(0) = 8$ $g(0) = 1$

$T_2(x) = 1 + 3(x-0) + \frac{8(x-0)^2}{2}$

$= 1 + 3x + 4x^2$

second derivative test

$f''(x)$ → ◡ ◠

ex: $y = 4x^4 + 3x^5 + 2x^3$

$y' = 16x^3 + 15x^4 + 6x^2$

$y'' = 48x^2 + 60x^3 + 12x$

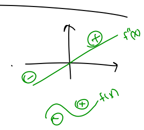
$= 12x(4x + 5x^2 + 1)$ lets pretend this is +

$= 0$

$x = 0$

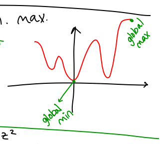
0
— + $f''(x)$
∩ ∪ $f(x)$

ex:



Absolute min. max.

find the value at the end points (y value)



ex: $h(z) = e^{z^2}$

$h'(z) = e^{z^2}(2z)$ → 0

critical points → $h'(x)$ → undefined, → 0

0
— + $f'(x)$
↘ ↗ $f(x)$

$h''(z) = e^{z^2}(2z)(2z) + (2)(e^{z^2})$

$= e^{z^2}(4z^2 + 2)$

+ + ∪ at all points

Absolute min and max in $[-1, 1]$

at 1 and −1 ⇝ global max

at 0 we have global min

z	$h(z)$
1	e
0	1
−1	e

LH rule

$\lim_{x \to \infty \text{ or } \to 0 \text{ or} \dots} f(x) \xrightarrow{\text{sub in}} \frac{0}{0} \quad \frac{\infty}{\infty} \quad \infty - \infty$

→ $0 \cdot \infty$ X

LH $\frac{f(x)}{g(x)}$ ⇝ $\frac{f'(x)}{g'(x)}$ not quotient rule

Ex. $\lim_{x \to 0} \frac{4x^2}{3x} \underset{0}{\overset{\text{sub in}}{\rightsquigarrow}} \frac{0}{0}$ X

$\xrightarrow{LH} \lim_{x \to 0} \frac{8x}{3} \underset{0}{\overset{\text{sub in}}{\rightsquigarrow}} \frac{0}{3} = 0$ ✓

Diff. equations:

ex: volume of a cell

$4\ \mu m^3/s$ ⇝ rate of change $V'(t)$

① $V'(t) = 4$

② Initial Conditions: at $t = 0$, $V = 100$

$V'(t) = 4 \longrightarrow V(t) = 4t + C$

$V(0) = 100$

$100 = 4(0) + C$

$\rightarrow C = 100$

$V(t) = 4t + 100$

Antiderivatives:

Antiderivatives

$\int f(x)\,dx$

ex: $\int 3x^2\,dx = \frac{3x^3}{3} + C$

1. substitution
2. by parts

ex: $\int \frac{2x}{1+x^2}\,dx$

$1+x^2 = u$

$2x\,dx = du \longrightarrow dx = \frac{du}{2x}$

$\int \frac{\cancel{2x}}{u} \cdot \frac{du}{\cancel{2x}} = \int \frac{1}{u}\,du$

$= \ln|u| + C$

$= \ln|\underbrace{1+x^2}_{+}| + C$

$= \ln(1+x^2) + C$

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ex: $\int 1.\ln x\,dx$

memorize

$\int u\,dv = u.v - \int v\,du$

$u = \ln x \longrightarrow du = 1/x$

$dv = 1 \longrightarrow v = x$

$\int \ln x = x\ln x - \int x \cdot 1/x\,dx$

$= x\ln x - \int 1\,dx$

$= x\ln x - x + C$

Euler's Method

$F'(x) =$

$f(x) \rightsquigarrow$ Euler's Method

$\Delta x \longrightarrow$ step size

$y_n = y_{n-1} + \underbrace{y'_{n-1}}_{\text{slope}} \Delta x$

IC $\rightarrow x_0$ and y_0 $\quad x=0 \;\hookrightarrow x=1$, $\Delta x = 0.5$

1. $y_1 = y_0 + y_0'\,\Delta x$ $\quad x=0,\ x=0.5,\ x=1$
2. $y_2 = y_1 + y_1'\,\Delta x \rightsquigarrow x_2$

Ex: $x_0 = 0 \quad y_0 = 1$ / $y(0)=1$, $f(0)=1$

$\Delta x = 0.25$

1. $x_1 = x_0 + \Delta x = 0.25$

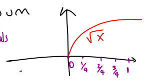
   $y_1 = y_0 + y_0'\,\Delta x$
2. $x_2 = x_1 + \Delta x = 0.25 + 0.25 = 0.5$

   $y_2 = y_1 + y_1'\,\Delta x$
3. $x_3 = x_2 + \Delta x = 0.75$

   $y_3 = y_2 + y_2'\,\Delta x$
4. $x_4 = x_3 + \Delta x = 1$

   $y_4 = y_3 + y_3'\,\Delta x$

Riemen's Sum



$[0, 1] \rightarrow$ 4 intervals

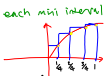
$\frac{1-0}{4} = 0.25$

$R_4 \rightsquigarrow$ right side point

$L_4 \rightsquigarrow$ left point

$M_4 \rightsquigarrow$ mid points of each mini interval

$\sqrt{x} \quad [0,1]$

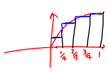


R_4

$\frac{1}{4}(\sqrt{1/4}) + \frac{1}{4}(\sqrt{2/4}) + \frac{1}{4}(\sqrt{3/4}) + \frac{1}{4}(\sqrt{1})$

M_4: for the first rectangle

$\frac{0 + 1/4}{2} = 1/8$



$y = \sqrt{1/8}$

$A = \sqrt{1/8} \times 1/4$

Integration by parts:

ex: $\int x \sin x\,dx$

$x = u \longrightarrow du = 1 \rightarrow x(-\cos x) - \int 1(-\cos x)\,dx$

$\sin x = dv \longrightarrow v = -\cos x$

$x = du \longrightarrow u = \frac{x^2}{2} \rightarrow (\sin x)(\frac{x^2}{2}) - \int \frac{x^2 \cos x}{2}\,dx$

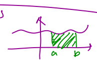
$\sin x = v \longrightarrow dv = \cos x$

wrong choice of u and v

$\int -\cos x\,dx = -\sin x + C$

$\int x \sin x\,dx = -x\cos x - (-\sin x) + C$

Definite Integrals



$\int_a^b f(x)\,dx$

$= F(x)\Big|_a^b = F(b) - F(a)$

Ex: $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

cancel out

let $\ln x = u$

$\rightarrow \frac{1}{x}dx = du \rightarrow dx = x\,du$

$\int \frac{x\,du}{x\sqrt{u}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C$

$\rightsquigarrow$ sub in u $\quad = 2\sqrt{\ln x}$

$\rightarrow$ calculate definite integral: $\left(2\sqrt{\ln x}\right)_e^{e^4}$

$= 2\sqrt{\ln e^4} - 2\sqrt{\ln e}$

$= 2\sqrt{4} - 2\sqrt{1}$

$= 4 - 2 = 2$

Diff. equations:

ex: Rate of change of a population is a function of time:

$p'(t) = t^2 + 3t + 4$

I.C. $P(0) = 1000$

$P(t) = ?$

$P(t) = \int P'(t)\,dt$

$P(t) = \int t^2 + 3t + 4\,dt$

$= \frac{t^3}{3} + \frac{3t^2}{2} + 4t + C$ general solution

makes it a unique solution

$P(0) = 1000$

$P(0) = \overbrace{\cancel{\frac{0}{3} + \frac{3}{2}(0)}}^{0} + 4(0) + C = 1000$

$C = 1000$

$P(t) = \frac{t^3}{3} + \frac{3t^2}{2} + 4t + 1000$

$y' = 3y$, IC $= 4$

$y = e^{3x} \longrightarrow y' = 3\overbrace{e^{3x}}^{y} = 3y$

$y' = \alpha y \xrightarrow[\text{Sol}^{\text{n}}]{\text{gen.}} y = e^{\alpha t}$

I.C. : $y(0) = V$

$y = \underbrace{\cancel{e^{\alpha(0)}}}_{1} k = V$

$k = V$

specific solution: $y = ke^{\alpha t}$

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ex: $\int \underset{\text{constant}}{b}(e^{-xt} + xe^{t})\,dt$

$= b\int e^{-xt} + xe^{t}\,dt$

$= b\left(\frac{e^{-xt}}{-x} + xe^{t}\right) + C$

ex: $\int b(e^{-xt} + xe^{t})\,dx$

$= b\int e^{-xt} + xe^{t}\,dx$

$= b\left(\frac{e^{-xt}}{-t} + \frac{x^2e^{t}}{2}\right) + C$

Implicit diff.

ex: $(1+x^2+y^2)^2 + y = 2e^x + 3$

(1) $(1+x^2+y^2)^2 + y - 2e^x - 3 = 0$

(2) $2(1+x^2+y^2)(2x+2yy') + y' - 2e^x - 0 = 0$

(3) Simplify if asked to find y'

$2(2x + 2x^3 + 2xy^2 + 2yy' + 2x^2yy' + 2y^3y') + y' - 2e^x =$

$= 4x + 4x^3 + 4xy^2 + 4yy' + 4x^2yy' + 4y^3y' + y' - 2e^x =$

$\rightarrow 4yy' + 4x^2yy' + 4y^3y' + y' = -4x - 4x^3 - 4xy^2 + 2e^x$

$y'(4y + 4x^2y + 4y^3 + 1) = -4x - 4x^3 - 4xy^2 + 2e^x$